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Distribution of Torque in a Dual-Frequency Liquid Crystal

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We derive the equations that relate the torque distribution to the voltage and configuration when voltages at two or more frequencies are superimposed in a liquid crystal for dual-frequency applications. In such a material, the dielectric anisotropy factor γ changes sign from positive to negative as the frequency is raised through a cross-over frequency f_c. The director torque is proportional to γ, to the square of the electric field E, and to $\sin(2\theta)$ where θ is the angle between the field and the director. The results are complicated by the fact that E is not constant across the cell, but varies with director orientation. The derived equations apply to the general case of a lossy medium. An example is given in which a high frequency overcomes the orienting effect of a low frequency.

1. INTRODUCTION

A liquid crystal for dual-frequency applications (dual-frequency material) is one in which the dielectric anisotropy $\epsilon_{\|} - \epsilon_{\perp}$ changes sign from positive to negative as the frequency is raised through a cross-over frequency f_c.[1,2] Below f_c, the electric field applies a torque that tends to align the liquid crystal director along the field direction, but above f_c, perpendicular to the field direction.

The torque is proportional to the dielectric anisotropy factor, to the square of the electric field, and to $\sin\theta\cos\theta$ where θ is the angle between the field and the director. As is well known, the field magnitude is not constant through the cell, but varies with director orientation. Thus, even at a single frequency, the torque varies in a nontrivial way across the cell. In a purely dielectric liquid crystal, one assumes that the normal component of the electric displacement

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vector **D** is constant because this gives **D** zero divergence, which is a fundamental relation that applies in the absence of space charge. An appropriate generalization of this relation for a lossy material is that the total current (conduction plus displacement) have zero divergence. The extreme case in which the conduction current alone has zero divergence has been considered by Deuling and Helfrich[3] and Deuling.[4]

In the external circuit, one controls only the voltage across the cell, not, of course, the field distribution. In this paper, we derive the equations that relate the torque distribution to the voltages and configuration when voltages at two or more frequencies are superimposed in a dual-frequency material.

2. CHARACTERIZATION OF THE MATERIAL

The dielectric and resistivity properties of a dual-frequency liquid crystal can be characterized by the dc conductivities $\sigma_{\parallel}^{dc}$, $\sigma_{\perp}^{dc}$, and frequency dependent complex permittivities $\epsilon_{\parallel}^{*}$ and $\epsilon_{\perp}^{*}$. These parameters can be determined from measurements of the dc resistivity, and of the capacitance and quality factor as a function of frequency, carried out in a homeotropically aligned cell for the $\parallel$ quantities and in a homogeneously aligned cell for the $\perp$ quantities.

We shall use asterisks to denote complex quantities.

Under electric field $E_{\parallel} = \text{Re}(E_{\parallel}^{*})$ the conduction current density is $J_{\parallel} = \sigma_{\parallel}^{dc}E_{\parallel}$ and the displacement current density is $\dot{D}_{\parallel} = \text{Re}(\epsilon_{\parallel}^{*}\dot{E}^{*})$. For a homeotropic cell of cross-sectional area A, the current in the circuit is $A(J_{\parallel} + \dot{D}_{\parallel})$. Hence, the admittance of such a cell of thickness d is

$$Y = \frac{A}{d}\left(\sigma_{\parallel}^{dc} + j\omega\epsilon_{\parallel}^{*}\right) = \frac{A}{d}\left(\sigma_{\parallel}^{dc} + \sigma_{\parallel} + j\omega\epsilon_{\parallel}\right) \tag{2.1}$$

where we have written

$$\epsilon_{\parallel}^{*} = \epsilon_{\parallel} - j\sigma_{\parallel}/\omega, \tag{2.2}$$

with $\epsilon_{\parallel} = \text{Re}(\epsilon_{\parallel}^{*})$, $\sigma_{\parallel} = \text{Re}(j\omega\epsilon_{\parallel}^{*})$. Eq. (2.1) is the same as the admittance of a parallel arrangement of a capacitance C_p and a resistance R_p having the values

$$C_p = \frac{\epsilon_{\parallel}A}{d}, \qquad R_p = \frac{d}{A\left(\sigma_{\parallel}^{dc} + \sigma_{\parallel}\right)}. \tag{2.3}$$

The quality factor Q is

$$Q = \omega C_p R_p = \frac{\omega \epsilon_{\parallel}}{\sigma_{\parallel}^{dc} + \sigma_{\parallel}}. \tag{2.4}$$

Similar equations apply to $\sigma_{\perp}^{dc}$ and $\epsilon_{\perp}$ in the homogeneously aligned cell. Thus, measurement of the dc conductivity and of C_p and Q as functions of frequency enable the material to be characterized.

3. ANALYSIS OF TORQUE

Dual-frequency materials typically have $\epsilon_{\perp}$ constant in the frequency range of interest while $\epsilon_{\parallel}$ drops with increasing frequency from a value $\epsilon^0 > \epsilon_{\perp}$ to a value of $\epsilon^{\infty} < \epsilon_{\perp}$. In a simple model with a single relaxation time, the complex permittivity is given by[5]

$$\epsilon_{\parallel}^* = \epsilon^{\infty} + \frac{\epsilon^0 - \epsilon^{\infty}}{1 + j\omega\tau}. \tag{3.1}$$

In what follows, we shall assume $\epsilon_{\perp}$ = constant but a general $\epsilon_{\parallel}^*$ as in (2.2).

If we replace $\epsilon_{\parallel}$ by $\epsilon_{\parallel}^*$, $\mathbf{D}$ by $\mathbf{D}^*$, and $\mathbf{E}$ by $\mathbf{E}^*$ in the relation between $\mathbf{D}$ and $\mathbf{E}$ for linear dielectrics,[6] we obtain

$$\mathbf{D}^* = \epsilon_{\perp}\left[\mathbf{E}^* + \gamma^*(\mathbf{E}^* \cdot \mathbf{n})\mathbf{n}\right] \tag{3.2}$$

$$\gamma^* = \left(\epsilon_{\parallel}^* - \epsilon_{\perp}\right)/\epsilon_{\perp} = \gamma - j\beta, \tag{3.3}$$

$$\gamma \equiv \mathrm{Re}(\gamma^*), \qquad \beta \equiv \mathrm{Re}(j\gamma^*) = \sigma_{\parallel}/\omega\epsilon_{\perp}. \tag{3.4}$$

If $\mathbf{E}$ has only a z component, given by

$$E_z = \mathrm{Re}(E^*), \tag{3.5}$$

then $\mathbf{D}$ has the components

$$D_z = \epsilon_{\perp}\mathrm{Re}\left[(1 + \gamma^* \cos^2\theta)E^*\right], \tag{3.6}$$

$$D_x = \epsilon_{\perp}\sin\theta\cos\theta\,\mathrm{Re}(\gamma^* E^*), \tag{3.7}$$

where θ is the angle between the director $\mathbf{n}$ and the z axis.

When the director orientation is not constant across the cell, one of the complications is that E_z varies with z through $\theta(z)$. Further, if the imaginary part of the permittivity is taken into account, there is a phase difference between D_z and E_z that depends on $\theta(z)$. This has the interesting consequence that E_z is not everywhere in phase with the applied voltage, but has a phase that varies with $\theta(z)$. To account for these z dependences, we take

$$E^* = Ee^{j(\omega t - \phi)} \tag{3.8}$$

where both E and ϕ are understood to depend on z through $\theta(z)$. Then (3.7) becomes

$$D_x = \epsilon_\perp E \sin\theta \cos\theta [\gamma \cos(\omega t - \phi) + \beta \sin(\omega t - \phi)]. \tag{3.9}$$

The torque per unit volume $\mathbf{P} \times \mathbf{E}$ has magnitude $D_x E_z$ and time average

$$\langle D_x E_z \rangle = \tfrac{1}{2}\gamma \epsilon_\perp E^2 \sin\theta \cos\theta. \tag{3.10}$$

The torque tends to align the director along the field (parallel or antiparallel) when $\gamma > 0$, and perpendicular to the field when $\gamma < 0$. In terms of the electric field amplitude E, the time average of the torque is given simply by (3.10). However, E itself has a complicated dependence on $\theta(z)$ that should be taken into account unless the anisotropy is small.

We turn now to the relation of E to the voltage and the configuration $\theta(z)$. The key to obtaining this relation is the condition that the total current $(\mathbf{J} + \dot{\mathbf{D}})$ have zero divergence. From the anisotropic form of Ohm's law, the z component of the conduction current is

$$J_z = \sigma_\perp^{dc}(1 + r\cos^2\theta) E_z, \tag{3.11}$$

$$r \equiv \left(\sigma_\parallel^{dc} - \sigma_\perp^{dc}\right)/\sigma_\perp^{dc}. \tag{3.12}$$

With J_z from (3.11), D_z from (3.6), and E_z from (3.5) and (3.8),

$$J_z + \dot{D}_z = \omega[p\cos(\omega t) + q\sin(\omega t)] \tag{3.13}$$

where

$$p = (a\cos\phi + b\sin\phi)E, \tag{3.14}$$

$$q = (a\sin\phi - b\cos\phi)E, \tag{3.15}$$

$$a = \sigma_\perp^{dc}(1 + r\cos^2\theta)/\omega + \epsilon_\perp \beta\cos^2\theta. \tag{3.16}$$

$$b = \epsilon_\perp (1 + \gamma\cos^2\theta). \tag{3.17}$$

The divergence-free nature of $\mathbf{J} + \dot{\mathbf{D}}$ requires p and q to be constant. This condition determines E and ϕ as functions of $\cos^2\theta$. The values of the constants p and q are then determined from the condition that the voltage $V\cos(\omega t)$ must be given by

$$V\cos(\omega t) = \int_{-d/2}^{d/2} E_z\, dz. \tag{3.18}$$

Define $\epsilon(z)$ and α by the equations

$$\epsilon(z) = (a^2 + b^2)^{1/2}, \tag{3.19}$$

$$\cos\alpha = a/\epsilon(z), \qquad \sin\alpha = b/\epsilon(z). \tag{3.20}$$

A brief excursion into algebra yields

$$E = (p^2 + q^2)^{1/2}/\epsilon(z), \tag{3.21}$$

$$E\cos\phi = (p\cos\alpha - q\sin\alpha)/\epsilon(z), \tag{3.22}$$

$$E\sin\phi = (p\sin\alpha + q\cos\alpha)/\epsilon(z). \tag{3.23}$$

In view of (3.18) and the form of these last equations, it proves expedient to define

$$I_1 \equiv \int_{-d/2}^{d/2} \frac{\sin\alpha}{\epsilon(z)}\, dz, \qquad I_2 \equiv \int_{-d/2}^{d/2} \frac{\cos\alpha}{\epsilon(z)}\, dz. \tag{3.24}$$

Substitution into (3.18) then yields

$$V\cos(\omega t) = (pI_2 - qI_1)\cos(\omega t) + (pI_1 + qI_2)\sin(\omega t), \tag{3.25}$$

whence

$$p = \frac{VI_2}{I_1^2 + I_2^2}, \qquad q = \frac{-VI_1}{I_1^2 + I_2^2}. \tag{3.26}$$

With p and q determined, (3.22) and (3.23) now enable the electric field to be found through (3.8). Note that the configuration $\theta(z)$ must be specified in order to evaluate the integrals I_1 and I_2.

By substitution from (3.26) into (3.13), we see that the admittance of the cell is

$$Y = \frac{AI_2}{I_1^2 + I_2^2} + \frac{jAI_1}{I_1^2 + I_2^2}, \tag{3.27}$$

which is like a parallel capacitance C and resistance R satisfying

$$C = AI_1/(I_1^2 + I_2^2), \qquad (3.28)$$

$$R = (I_1^2 + I_2^2)/AI_2. \qquad (3.29)$$

The Q of this RC combination is

$$Q = \omega CR = I_1/I_2. \qquad (3.30)$$

Eqs. (3.30) and (3.28) can be solved for I_1 and I_2, making it possible to express I_1 and I_2 in terms of experimentally measured values C and Q, as an alternative to carrying out the integrations in (3.24). The results are

$$I_2 = \frac{AQ}{(Q^2+1)C}, \qquad I_1 = \frac{AQ^2}{(Q^2+1)C}. \qquad (3.31)$$

In the nondissipative approximation, $a = 0$, $\alpha = \pi/2$, $I_2 = 0$, and I_1 reduces to

$$I_1 = \int_{-d/2}^{d/2} \frac{dz}{\epsilon(z)} \qquad \text{(lossless case)} \qquad (3.32)$$

so that the capacitance is A/I_1, in agreement with (3.28).

Returning now to the torque (3.10), we use E from (3.21), and p and q from (3.26) to obtain

$$\langle D_x E_z \rangle = \frac{V^2 \epsilon_\perp \gamma \sin\theta \cos\theta}{2(I_1^2 + I_2^2)(\epsilon(z))^2}. \qquad (3.33)$$

If voltages at two or more frequencies are superimposed, the resultant voltage can be expressed as

$$V(t) = \sum_i V_i \cos(\omega_i t). \qquad (3.34)$$

Associated with each frequency ω_i is an electric field distribution $E_i(z)$, $\phi_i(z)$ and corresponding components of electric displacement. The instantaneous torque is a complicated product of sums: $(\Sigma_i D_{xi})[\Sigma_i E_i(z)]$. But because of the orthogonality properties of $\sin(\omega t)$ and $\cos(\omega t)$, its time average reduces to a sum of terms like Eq. (3.33).

Thus,

$$\langle D_x E_z \rangle = \tfrac{1}{2}\epsilon_\perp \sin\theta\cos\theta \sum_i \gamma_i [E_i(z)]^2 \tag{3.35}$$

where

$$E_i(z) = \frac{V_i}{(I_1^2 + I_2^2)_i^{1/2}\epsilon_i(z)}. \tag{3.36}$$

With I_1 and I_2 from (3.31), Eq. (3.36) becomes

$$E_i(z) = \frac{V_i C_i (1 + Q_i^2)^{1/2}}{A Q_i \epsilon_i(z)}. \tag{3.37}$$

In a dual-frequency material, the voltage of a high frequency having γ_i negative and of a low frequency having γ_i positive can be chosen such that their torques nearly cancel each other.

If the voltage is applied as a square wave or some shape other than sinusoidal, then the above equations apply to each Fourier component.

4. EXAMPLE OF TORQUE DISTRIBUTION

In the model (3.1),

$$\gamma_i = \gamma^\infty + \frac{\gamma^0 - \gamma^\infty}{1 + \omega_i^2\tau^2}. \tag{4.1}$$

We suppose the configuration is given by

$$\theta(z) = \pi/2 + (\pi - 2\theta_b)z/d, \tag{4.2}$$

which is an exact solution for the symmetric horizontal equilibrium configuration at zero field in the special case of equal elastic constants. Then

$$I_2 = \frac{d}{\epsilon_\perp(\pi - 2\theta_b)} \int_{\theta_b}^{\pi-\theta_b} \frac{\beta\cos^2\theta\, d\theta}{(1 + \gamma\cos^2\theta)^2 + (\beta\cos^2\theta)^2}, \tag{4.3}$$

$$I_1 = \frac{d}{\epsilon_\perp(\pi - 2\theta_b)} \int_{\theta_b}^{\pi-\theta_b} \frac{(1 + \gamma\cos^2\theta)\, d\theta}{(1 + \gamma\cos^2\theta)^2 + (\beta\cos^2\theta)^2}. \tag{4.4}$$

The substitution $x = \cos^2\theta$ brings these integrals to

$$I_2 = \frac{d}{\epsilon_\perp(\pi - 2\theta_b)} \int_0^{\cos^2\theta_b} \frac{\beta x\, dx}{\left[(1+\gamma x)^2 + (\beta x)^2\right]\left[x(1-x)\right]^{1/2}}, \tag{4.5}$$

$$I_1 = \frac{d}{\epsilon_\perp(\pi - 2\theta_b)} \int_0^{\cos^2\theta_b} \frac{(1+\gamma x)\, dx}{\left[(1+\gamma x)^2 + (\beta x)^2\right]\left[x(1-x)\right]^{1/2}}. \tag{4.6}$$

These integrals can be handled by partial fraction expansions of the rational factor in the integrands.

In the nondissipative case, $I_2 = 0$ and I_1 becomes

$$I_1 = \frac{2d\left\{\frac{\pi}{2} - \tan^{-1}\left[\tan\theta_b/(1+\gamma)^{1/2}\right]\right\}}{\epsilon_\perp(\pi - 2\theta_b)(1+\gamma)^{1/2}}. \tag{4.7}$$

The capacitance of the symmetric horizontal configuration is then $C = A/I_1$.

The parameters in Eqs. (3.1) or (4.1) can be chosen to fit the dielectric properties of the material of Ref. 1, Table I at 0.1, 5.1, and 40 kHz. The results are shown in Table I. The values of $\bar{\epsilon}$, γ, and β *calculated from the model* (3.1) at 0.1 kHz and 40 kHz are shown in Table II. Since the values of β are relatively small, we shall carry out the example based on the lossless approximation. With $\bar{\epsilon} \equiv d/I_1$, and $I_2 = 0$, Eq. (3.36) reduces to

$$E_i(z) = \frac{V_i\bar{\epsilon}_i}{\epsilon_\perp d\left[1 + \gamma_i\cos^2\theta(z)\right]}. \tag{4.8}$$

TABLE I

Parameters in Eqs. (3.1) and (4.1)
($\epsilon_0 = 8.854 \times 10^{-12}$ F/m)

Parameter	Value
$\epsilon_\perp/\epsilon_0$	5.25
ϵ^0/ϵ_0	8.20
$\epsilon^\infty/\epsilon_0$	3.40
τ	39.4 μsec
γ^0	0.562
γ^∞	−0.352

TABLE II

Values at 0.1 kHz and 40 kHz with $\theta_b = 70°$

Parameter	0.1 kHz	40 kHz
β	0.02	0.09
γ	0.562	-0.343
$\bar{\epsilon} = d/I_1$	$5.365\epsilon_0$	$5.178\epsilon_0$

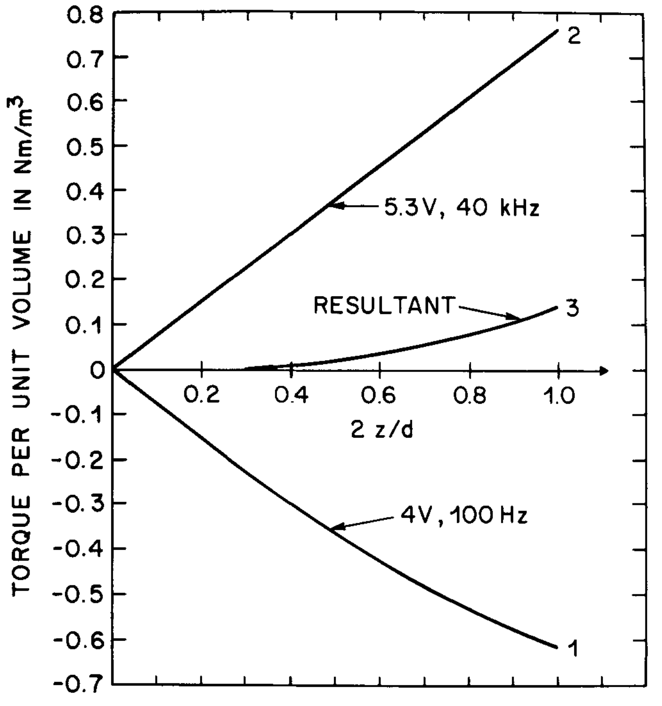

FIGURE 1 Examples of torque distribution in the symmetric horizontal configuration of Eq. (4.2) with $\theta_b = 70°$ and $d = 10\ \mu$m. The curves show, for the material of Tables I and II, the time averaged torque per unit volume versus $2z/d$ resulting from (1) 4 V amplitude at 100 Hz, (2) 5.3 V amplitude at 40 kHz, and (3) the above combination. In this example, the high frequency overcomes the orienting effect of the low frequency.

Figure 1 shows the distribution of torque per unit volume for V_1 = 4 V at 0.1 kHz and V_2 = 5.3 V at 40 kHz, calculated from (3.35) with $\theta(z)$ from (4.2) and $E_i(z)$ from (4.8).

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